

# Evolution Strategies in Optimization Problems\*

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## Abstract

Evolution Strategies are inspired in biology and part of a larger research field known as Evolutionary Algorithms. Those strategies perform a random search in the space of admissible functions, aiming to optimize some given objective function. We show that simple evolution strategies are a useful tool in optimal control, permitting to obtain, in an efficient way, good approximations to the solutions of some recent and challenging optimal control problems.

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**Keywords.** Random search, Monte Carlo method, evolution strategies, optimal control, discretization.

## 1 Introduction

Evolution Strategies (ES) are algorithms inspired in biology, with publications dating back to 1965 by separate authors H.P. Schwefel and I. Rechenberg (cf. [5]). ES are part of a larger area called Evolutionary Algorithms that perform a random search in the space of solutions aiming to optimize some objective function. It is common to use biological terms to describe these algorithms. Here we make use of a simple ES algorithm known as the  $(\mu, \lambda)$ -ES method [5], where  $\mu$  is the number of *progenitors* and  $\lambda$  is the number of generated approximations, called *offsprings*. Progenitors are *recombined* and *mutated* to produce, at each generation,  $\lambda$  *offsprings* with innovations sampled from a multivariate normal distribution. The variance can also be subject to mutation, meaning that it is part of the *genetic code* of the population. Every solution is evaluated by the objective function and one or some of them selected to be the next *progenitors*,

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allowing the search to go on, stopping when some criteria is met. In this paper we use a recent convergence result proved by A. Auger in 2005 [3]. The log-linear convergence is achieved for the optimization problems we investigate here, and depends on the number  $\lambda$  of search points.

Usually optimal control problems are approximately solved by means of numerical techniques based on the gradient vector or the Hessian matrix [2]. Compared with these techniques, ES provide easier computer coding because they only use measures from a discretized objective function. A first work combining these two research fields (ES and optimal control) was done by D.S. Szarkowicz in 1995 [14], where a Monte Carlo method (an algorithm with the same principle as ES) is used to find an approximation to the classical brachistochrone problem. In the late nineties of the XX century, B. Porter and his collaborators showed how ES are useful to synthesize optimal control policies that minimize manufacturing costs while meeting production schedules [8]. The use of ES in Control has grown during the last ten years, and is today an active and promising research area. Recent results, showing the power of ES in Control, include Hamiltonian synthesis [11], robust stabilization [10], and optimization [4]. Very recently, it has also been shown that the theory of optimal control provides insights that permit to develop more efective ES algorithms [1].

In this work we are interested in two classical problems of the calculus of variations: the 1696 brachistochrone problem and the 1687 Newton's aerodynamical problem of minimal resistance (see e.g. [15]). These two problems, although classical, are source of strong current research on optimal control and provide many interesting and challenging open questions [7, 13]. We focus our study on the brachistochrone problem with restrictions proposed by A.G. Ramm in 1999 [12], for which some questions still remain open (see some conjectures in [12]); and on a generalized aerodynamical minimum resistance problem with non-parallel flux of particles, recently studied by Plakhov and Torres [7, 16]. Our results show the effectiveness of ES algorithms for this class of problems and motivate further work in this direction in order to find the (yet) unknown solutions to some related problems, as the ones formulated in [6].

## 2 Problems and Solutions

All the problems we are interested in share the same formulation:

$$\min T[y(\cdot)] = \int_{x_0}^{x_f} L(x, y(x), y'(x)) dx$$

on some specified class of functions, where  $y(\cdot)$  must satisfy some given boundary conditions  $(x_0, y_0)$  and  $(x_f, y_f)$ .

We consider a simplified  $(\mu, \lambda)$ -ES algorithm where we put  $\mu = 1$ , meaning that on each generation we keep only one *progenitor* to generate other candidate solutions, and set  $\lambda = 10$  meaning we generate 10 candidate solutions called *offsprings* (this value appear as a reference value in the literature). Also, the algorithm uses an individual and constant  $\sigma^2$  variance on each coordinate, which

is fixed to a small value related with the desired precision. The number of iterations was 100 000 and  $\sigma^2$  was tuned for each problem. We got convergence in useful time. The simplified  $(1, 10)$ –ES algorithm goes as follow:

1. Set an equal spaced sequence of  $n$  points  $\{x_0, \dots, x_i, \dots, x_f\}$  where  $i = 1, \dots, n - 2$ ;  $x_0$  and  $x_f$  are kept fixed (given boundary conditions);
2. Generate a randomly piecewise linear function  $y(\cdot)$  that approximate the solution, defined by a vector  $y = \{y_0, \dots, y_i, \dots, y_f\}$ ,  $i = 1, \dots, n - 2$ ; transform  $y$  in order to satisfy the boundary conditions  $y_0$  and  $y_f$  and the specific problem restrictions on  $y$ ,  $y'$  or  $y''$ ;
3. Do the following steps a fixed number  $N$  of times:
  - (a) based on  $y$  find  $\lambda$  new candidate solutions  $Y^c$ ,  $c = 1, \dots, \lambda$ , where each new candidate is produced by  $Y^c = y + N(0, \sigma^2)$  where  $N(0, \sigma^2)$  is a vector of random perturbations from a normal distribution; transform each  $Y^c$  to obey boundary conditions  $y_0$  and  $y_f$  and other problem restrictions on  $y$ ,  $y'$  or  $y''$ ;
  - (b) determine  $T^c := T[Y^c]$ ,  $c = 1, \dots, \lambda$ , and choose the new  $y := Y^c$  as the one with minimum  $T^c$ .

In each iteration the best solution must be kept because  $(\mu, \lambda)$ –ES algorithms don't keep the best solution from iteration to iteration.

The next subsections contain a description of the studied problems, respective solutions and the approximations found by the described algorithm.

## 2.1 The classical brachistochrone problem, 1696

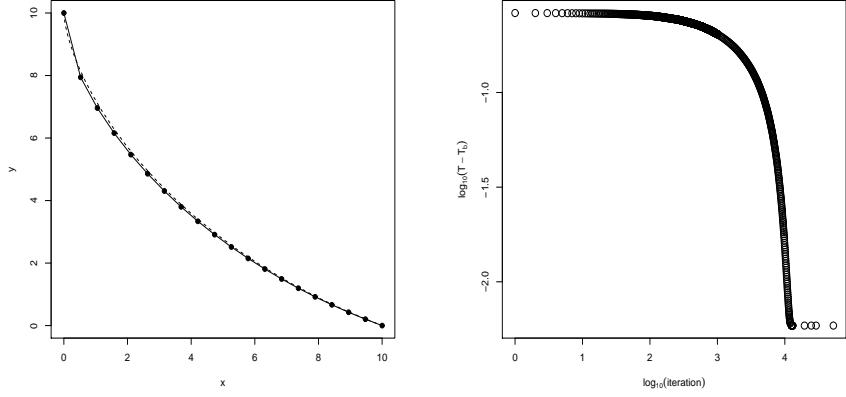
**Problem statement.** The brachistochrone problem consists in determining the curve of minimum time when a particle starting at a point  $A = (x_0, y_0)$  of a vertical plan goes to a point  $B = (x_1, y_1)$  in the same plane under the action of the gravity force and with no initial velocity. According to the energy conservation law  $\frac{1}{2}mv^2 + mgy = mgy_0$  one easily deduce that the time a particle needs to reach  $B$  starting from point  $A$  along curve  $y(\cdot)$  is given by

$$T[y(\cdot)] = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \sqrt{\frac{1 + (y')^2}{y_0 - y}} dx \quad (1)$$

where  $y(x_0) = y_0$ ,  $y(x_1) = y_1$ , and  $y \in C^2(x_0, x_1)$ . The minimum to (1) is given by the famous Cycloid:

$$\gamma : \begin{cases} x = x_0 + \frac{a}{2}(\theta - \sin \theta) \\ y = y_0 - \frac{a}{2}(1 - \cos \theta) \end{cases}$$

with  $\theta_0 \leq \theta \leq \theta_1$ ,  $\theta_0$  and  $\theta_1$  the values of  $\theta$  in the starting and ending points  $(x_0, y_0)$  and  $(x_1, y_1)$ . The minimum time is given by  $T = \sqrt{a/(2g)}\theta_1$ , where parameters  $a$  and  $\theta_1$  can be determined numerically from boundary conditions.



(a) Continuous line with dots is the piecewise approximate solution; the dashed line is the optimal solution.  
 (b) Logarithm of iterations vs. logarithmic distance to the minimum value of (1)

Figure 1: The brachistochrone problem and approximate solution.

**Results and implementation details.** Consider the following three curves and the correspondent time a particle needs to go from  $A$  to  $B$  through them:

$T_b$ : The brachistochrone for the problem with  $(x_0, x_1) = (0, 10)$ ,  $(y_0, y_1) = (10, 0)$  has parameters  $a \simeq 5.72917$  and  $\theta_1 = 2.41201$ ; the time is  $T_b \simeq 1.84421$ ;

$T_{es}$ : A piecewise linear function with 20 segments shown in fig. 1(a) was found by ES; the time is  $T_{es} = 1.85013$ ;

$T_o$ : A piecewise linear function with 20 segments defined over the Brachistochrone; the time is  $T_o = 1.85075$ .

From fig. 1(a) one can see that the piecewise linear solution is made of points that are not over the brachistochrone because that is not the best solution for piecewise functions. We use  $\sigma = 0.01$  (see appendix for cpu-times). Fig. 1(b) shows that a little more than 10 000 iterations are needed to reach a good solution for the 20 line segment problem.

## 2.2 Brachistochrone problem with restrictions, 1999

**Problem statement.** Ramm (1999) [12] presents a conjecture about a brachistochrone problem over the set  $S$  of convex functions  $y$  (with  $y''(x) \geq 0$  a.e.) and  $0 \leq y(x) \leq y_0(x)$ , where  $y_0$  is a straight line between  $A = (0, 1)$  and  $B = (b, 0)$ ,  $b > 0$ . Up to a constant, the functional to be minimized is formulated as in (1):

$$T[y(\cdot)] = \int_0^b \frac{\sqrt{1 + (y')^2}}{\sqrt{1 - y}} dx.$$

Let  $P$  be the line connecting  $AO$  and  $OB$ , where  $O = (0, 0)$ ;  $P_{br}$  be the polygonal line connecting  $AC$  and  $CB$ ,  $C = (\pi/2, 0)$ . Then,  $T_0 := T(y_0) = 2\sqrt{1+b^2}$ ,  $T_P := T(P) = 2+b$ ,  $T(P_{br}) = \sqrt{4+\pi^2} + b - \pi/2$ . Let the brachistochrone be  $y_{br}$ . The following inequalities, for each  $y \in S$ , hold [12]:

1. if  $0 < b < 4/3$  then  $T(y_{br}) \leq T(y) < T_P$ ;
2. if  $4/3 \leq b \leq \pi/2$  then  $T(y_{br}) \leq T(y) \leq T_0$ ;
3. if  $b > \pi/2$  then  $T(P_{br}) < T(y) \leq T_0$ .

The classical brachistochrone solution holds for cases 1 and 2 only. For the third case, Ramm has conjectured that the minimum time curve is composed by the brachistochrone between  $(0, 1)$  and  $(\pi/2, 0)$  and then by the horizontal segment between  $(\pi/2, 0)$  and  $(x_f, 0)$ .

**Results and implementation details.** We study the problem with  $b = 2$ . Our results give force to Ramm's conjecture mentioned above for case 3. We compare three descendant times:

$T_{br}$ : The conjectured solution in continuous time takes  $T_{br} = \sqrt{\alpha/9.8}\theta_f + (b - \pi/2)/\sqrt{2*9.8} = 0.8066$ ;

$T_{es}$ : The 20 segment piecewise linear solution found by ES needs  $T_{es} = 0.8107$ ;

$T_o$ : The 20 segment piecewise linear solution with points over the conjectured solution needs  $T_o = 0.8111$ .

Previous values and fig. 2 permits to take similar conclusions than the ones obtained for the pure brachistochrone problem (§2.1). We use  $\sigma = 0.001$  (see appendix for cpu-times). Fig. 2(b) shows that less than 10 000 iterations are needed to reach a good solution.

### 2.3 Newton's minimum resistance, 1687

**Problem statement.** Newton's aerodynamical problem consists in determining the minimum resistance profile of a body of revolution moving at constant speed in a rare medium of equally spaced particles that don't interact with each other. Collisions with the body are assumed to be perfectly elastic. Formulation of this problem is: minimize

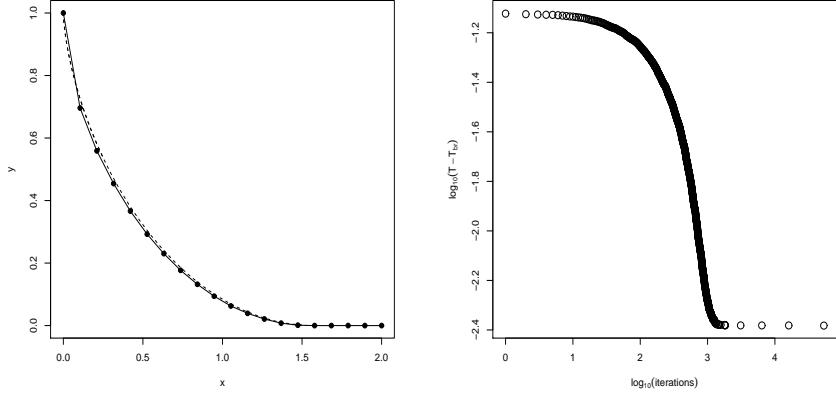
$$R[y(\cdot)] = \int_0^r \frac{x}{1 + \dot{y}(x)^2} dx$$

where  $0 \leq x \leq r$ ,  $y(0) = 0$ ,  $y(r) = H$  and  $y'(x) \geq 0$ . The solution is given in parametric form:

$$x(u) = 2\lambda u, \quad y(u) = 0, \quad \text{for } u \in [0, 1];$$

$$x(u) = \frac{\lambda}{2} \left( \frac{1}{u} + 2u + u^3 \right), \quad y(u) = \frac{\lambda}{2} \left( -\log u + u^2 + \frac{3}{4}u^4 \right) - \frac{7\lambda}{8}, \quad \text{for } u \in [1, u_{\max}].$$

Parameters  $\lambda$  and  $u_{\max}$  are obtained solving  $x(u_{\max}) = r$  and  $y(u_{\max}) = H$ .



(a) Continuous line with dots is the obtained approximated solution; dashed line Ramm's conjectured solution.  
(b) Logarithm of iterations vs. logarithmic distance to minimum integral value.

Figure 2: Ramm's conjectured solution and approximate solution.

**Results and implementations details.** For  $H = 2$  we have:

$R_{newton}$ : The exact solution has resistance  $R_{newton} = 0.0802$ ;

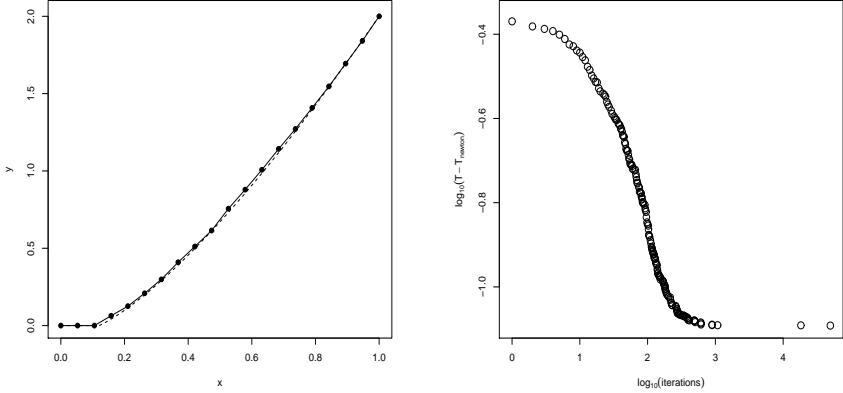
$R_{es}$ : The 20 segment piecewise linear solution found by ES has  $R_{es} = 0.0809$ ;

$R_o$ : The 20 segment piecewise linear solution with points over the exact solution leads to  $R_o = 0.0808$ .

Newton's problem reveals to be more complex than previously studied brachistochrone problems. Trial-and-error was needed in order to find a useful  $\sigma^2$  value. For example, using  $\sigma = 0.001$  our algorithm seems to stop in some local minimum. In fig. 3 an approximate solution with  $\sigma = 0.01$  is shown. We also have observed that changing the starting point causes minor differences in the approximate solution. The achieved ES solution should be better since  $R_o$  is better than  $R_{es}$ . One possible explanation for this fact is that we are using 20  $x_i$  fixed points and the optimal solution has a break point at  $x = 2\lambda$ . We use  $\sigma = 0.01$  (see appendix for cpu-times). Fig. 3(b) shows that less than 1000 iterations are needed to reach a good solution.

## 2.4 Newton's problem with temperature, 2005

**Problem statement.** The problem consists in determining the body of minimum resistance, moving with constant velocity in a rarefied medium of chaotically moving particles with velocity distributions assumed to be radially symmetric in the Euclidian space  $\mathbb{R}^d$ . This problem was posed and solved in 2005-2006 by Plakhov and Torres [7, 16]. It turns out that the two-dimensional



(a) Continuous line with dots is the obtained approximation; the dashed line the optimal solution. (b) Logarithm of iterations vs. logarithmic distance to minimum integral value.

Figure 3: Optimal solution to Newton's problem and approximation.

problem ( $d = 2$ ) is more richer than the three-dimensional one, being possible five types of solutions when the velocity of the moving body is not 'too slow' or 'too fast' compared with the velocity of particles.

The pressure at the body surface is described by two functions: in the front of the body the flux of particles causes resistance, in the back the flux causes acceleration. We consider functions found in [16], where the two flux functions  $p_+$  and  $p_-$  are given by  $p_+(u) = \frac{1}{1+u^2} + 0.5$  and  $p_-(u) = \frac{0.5}{1+u^2} - 0.5$ . We also consider a body of fixed radius 1. The optimal solution depends on the body height  $h$ : the front solution is denoted by  $f_{h+}$ , which depends on some appropriate front height  $h_+$ ; and the solution for the rear is denoted by  $f_{h-}$ , depending on some appropriate height  $h_-$ . Optimal solutions  $f_{h+}$  and  $f_{h-}$  are obtained:

$$f_{h+} = \min_{f_h} \mathbb{R}_+(f_h) = \int_0^1 p_+(f'_h(t)) dt$$

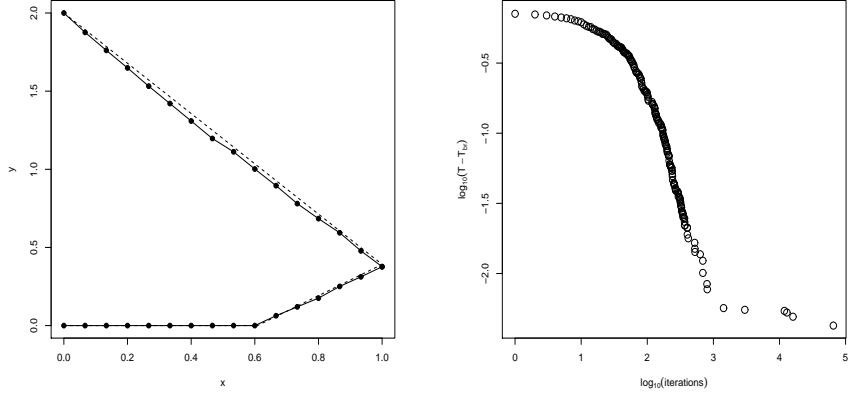
and

$$f_{h-} = \min_{f_h} \mathbb{R}_-(f_h) = \int_0^1 p_-(f'_h(t)) dt.$$

Then, the body shape is determined by minimizing

$$R(h) = \min_{h_+ + h_- = h} (\mathbb{R}_+(f'_{h+}) + \mathbb{R}_-(f'_{h-})).$$

Solution can be of five types ( $d = 2$ ). From functions  $p_+$  and  $p_-$  one can determine constants  $u_+^0$ ,  $u_*$ ,  $u_-^0$  and  $h_-$ . Then, depending on the choice of the height  $h$ , theory developed in [7, 16] asserts that the minimum resistance body is:



(a) Continuous line with dots is the obtained approximation; dashed line the optimum. (b) Logarithm of iterations vs. logarithmic distance to minimum integral value.

Figure 4: 2D Newton-type problem with temperature.

1. a trapezium if  $0 < h < u_+^0$ ;
2. an isosceles triangle if  $u_+^0 \leq h \leq u_*$ ;
3. the union of a triangle and a trapezium if  $u_* < h < u_* + u_-^0$ ;
4. if  $h \geq u_* + u_-^0$  the solution depends on  $h_-$  and can be a union of two isosceles triangles with common base with heights  $h_+$  and  $h_-$  or the union of two isosceles triangles and a trapezium;
5. a combination of a triangle, trapezium and other triangle, depending on some other particular conditions (cf. [7]).

**Results and implementation details.** We illustrate the use of ES algorithms for  $h = 2$ . Following section 4.1 of [7] we have  $u_* \simeq 1.60847$  and  $u_-^0 = 1$ , so this is case 3 above:  $u_* < h < u_* + u_-^0$ . The resistance values are:

$R_{pd}$ : The exact solution has resistance  $R_{pd} = 0.681$ ;

$R_{es}$ : The 31 segment piecewise linear solution found by ES has  $R_{es} = 0.685$ ;

Similar to the classical problem of Newton (§2.3), some hand search for the parameter  $\sigma^2$  was needed. We use  $\sigma = 0.01$  and piecewise approximation with 31 equal spaced segments in  $xx$  (see appendix for cpu-times). Fig. 3(b) shows that only little more than 1 000 iterations are needed to reach a good solution.

### 3 Conclusions and future directions

Our main conclusion is that a simple ES algorithm can be effectively used as a tool to find approximate solutions to some optimization problems. In the present work we report simulations that motivate the use of ES algorithms to find good approximate solutions to brachistochrone-type and Newton-type problems. We illustrate our approach with the classical problems and with some recent and still challenging problems. More precisely, we considered the 1696 brachistochrone problem (B); the 1687 Newton's aerodynamical problem of minimal resistance (N); a recent brachistochrone problem with restrictions (R) studied by Ramm in 1999, and where some open questions still remain [12]; and finally a generalized aerodynamical minimum resistance problem with non-parallel flux of particles (P), recently studied by Plakhov and Torres [7, 16] and which gives rise to other interesting questions [6].

We argue that the approximated solutions we have found by the ES algorithm are of good quality. We give two reasons. First, for the Brachistochrone and Ramm's problems the functional value for the ES approximation was better than the linear interpolation over the exact solution, showing that ES algorithm is capable of a good precision. The second reason is the low relative error  $r(T_Y, T_y)$  between the functional over the exact solution  $T_y$  and the approximate solution  $T_Y$ , as shown in the following table:

Pr.	$\max  Y_k - y_k $	$r(T_Y, T_y)$	Pr.	$\max  Y_k - y_k $	$r(T_Y, T_y)$
(B)	0.15	0.001	(N)	0.08	0.01
(R)	0.09	0.003	(P)	0.07	0.001

where  $y_k$  are points over the exact solution of the problem and  $Y_k$  are points from the piecewise approximation. We note that  $\max |Y_k - y_k|$  need not to be zero because the best continuous solution and the best linear solution cannot be superposed.

ES algorithms use computers in an intensive way. For brachistochrone-type and Newton-type problems, and nowadays computing power, few minutes of simulation (or less) were enough on an interpreted language (see appendix).

More research is needed to tune this kind of algorithms and obtain more accurate solutions. Special attention must be put in qualifying an obtained ES approximation: Is it a minimum of the energy function? Is it local or global? Another question is computer efficiency. Waiting few minutes in recent computers is not bad, but can we improve the running times?

Concerning the accuracy, several new ES algorithms have been proposed. These algorithms can tune  $\sigma$  values and use generated second order information that can influence the precision and time needed. Also the use of random  $xx$  points (besides  $y$  piecewise linear solution) should be investigated.

We believe that the simplicity of the technique considered in the present work can help in the search of solutions to some open problems in optimal control. This is under investigation and will be addressed elsewhere.

## Appendix – hardware and software

The code developed for this work can be freely obtained from the first author's web page, at <http://www.mat.ua.pt/jpedro/evolution/>.

In most of our investigations few minutes were sufficient for getting a good approximation for all the considered problems, even using a code style prone to humans rather than machines (code was done concerning clearness of concepts rather than execution speed). Our simulations used a Pentium 4 CPU 3 GHz, running Debian Linux <http://www.debian.org>. The language was R [9], chosen because it is a fast interpreted language, numerically oriented to statistics and freely available.

The following CPU-times were obtained with command

```
time R CMD BATCH problem.R
```

where `time` keeps track of cpu used and `R` calls the interpreter. The times are rounded and the last column estimates the time for a first good solution:

Problem	Section	100 000 iterations	'Good solution' at
Brachistochrone	§2.1	10 min	1 min
Ramm conjecture	§2.2	10 min	1 min
Newton	§2.3	9 min	10 sec
Plakhov & Torres	§2.4	14 min	10 sec

We note that the per iteration 'step' was  $\sigma = 0.001$  in the brachistochrone(-type) problems and  $\sigma = 0.01$  for the Newton(-type) problems. Using a compiled language like C one can certainly improve times by several orders of magnitude.

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